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AN ANALYTICAL DETERMINATION OF TEMPERATURE OSCILLATIONS IN A WALL HEATED BY ALTERNATING CURRENT

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SUMMARY

A solution is obtained for a complete temperature history and profile of a sinusoidally heated flat plate with constant thermal and electrical properties and insulated at one surface. The magnitudes of the uniform temperature oscillations at both surfaces of the plate, caused by the sinusoidal heat source, are calculated from the solution by a digital computer and are presented in a graphical form as functions of frequency and heat-transfer parameters.

The analysis is valid for thin, long tubes heated by alternating current. Magnitudes of uniform surface temperature oscillations are given in terms of heat flux in a table form, covering the range of variables pertinent to the boiling-heat-transfer field. It is shown that, at high heat fluxes, alternating-current heating can result in large surface temperature variations.

INTRODUCTION

The emphasis in recent heat-transfer investigations has been on achieving higher heat fluxes. An example is the field of boiling heat transfer where heat fluxes in excess of 10^7 Btu/(hr)(sq ft) have been obtained by passing high electrical currents through test elements. Electrical heating is commonly used because of the general ease and accuracy with which the electrical power may be controlled and recorded.

In heat-transfer studies, some knowledge of the temperature of the heat-transfer surface is required. This has been accomplished with varying degrees of success by measuring the heat-transfer surface temperature directly or, more commonly, by measuring the temperature of some point within the wall and then computing the temperature drop through the wall.

Direct electric current will produce a constant heat generation within the test-section wall, while alternating current will result in a uniform, but periodically oscillating, heat source. This periodic heat source in turn causes a periodic or cyclic temperature variation in the wall. Many temperature recorders are sensitive only to frequencies much lower than those of the readily available alternating current, giving only mean temperatures and not the temperature variations due to the oscillatory heating source. However, at high heat fluxes these variations may become of large magnitude and therefore have an important effect on the heat-transfer process.

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That neglecting the wall temperature variations resulting from alternating-current heating could result in an unexpected error was first suggested in reference 1, in which an analysis was made of a sinusoidally heated flat plate. Using the Laplace transformation method, the authors obtained a transfer function, the real part of which was used to determine the amplitudes of the uniform temperature oscillations. Their solution[1] is in a closed form for the amplitudes of the uniform temperature variations, but does not describe the temperature history.

Bertolotti, et al. (ref. 2) in a recent boiling study showed that alternating-current heating could appreciably affect experimental results. Their conclusion was based on an analysis incorporating the assumption of infinite conductivity of the wall, which resulted in a simplified solution for the amplitudes of the uniform temperature fluctuations. The assumption was not substantiated.

An expression is derived herein for the complete temperature history and profile of a sinusoidally heated flat plate with constant thermal and electrical properties and with one insulated surface. From this solution the magnitudes of the uniform temperature oscillations due to alternating-current heating were calculated by means of an IBM 704 digital computer. These magnitudes are presented in graphical form as functions of a dimensionless frequency and a heat-transfer parameter (Biot modulus). A comparison is made between the magnitudes obtained herein and those reported in reference 1, and the validity of the assumption used in reference 2 is tested.

ANALYSIS

Assumptions and Solution

When alternating current is passed through the walls of a tube, the heat generated per unit volume per unit time is, by Ohm's law,

[1]The constant B defined in the solution of ref. 1 (eq. III-A-18) should read

$$B = \left[\sqrt{\omega}(\cosh \sqrt{\omega} \sin \sqrt{\omega} + \sinh \sqrt{\omega} \cos \sqrt{\omega}) + h \sinh \sqrt{\omega} \sin \sqrt{\omega}\right]$$

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$$q''' = \frac{i^2 R}{\pi\left(r_o^2 - r_i^2\right)L}$$

(All symbols are defined in appendix A.) The imposed sinusoidal (i.e., alternating) current i is given by

$$i = \sqrt{2}\, I_{eff} \sin \omega\tau$$

where I_{eff} is equal in magnitude to the steady current that would cause the same average heating as the varying current actually existing. Hence,

$$q''' = 2I_{eff}^2 R \frac{\sin^2 \omega\tau}{\pi\left(r_o^2 - r_i^2\right)L} \tag{1}$$

By defining the mean heat-transfer area of the tube as $\pi(r_o + r_i)L$ and since $q = I_{eff}^2 R$, equation (1) reduces to

$$q''' = \frac{2}{l}\left(\frac{q}{A}\right) \sin^2 \omega\tau \tag{2}$$

If the ratio of the outer to inner radii of the tube is smaller than 1.4, the tube may be represented by a flat plate with a resulting error of less than 1 percent (ref. 3). Hence, consider a flat plate with unidirectional heat flow (negligible end effects), insulated at one surface and rejecting heat at the other. The plate is heated by alternating current and is assumed to have constant thermal and electrical properties. Thus, the differential equation for the temperature of the plate is:

$$\rho c_p \frac{\partial t}{\partial \tau} = k \frac{\partial^2 t}{\partial x^2} + q''' \tag{3}$$

Substituting equation (2) in equation (3) and rearranging yield

$$\frac{\partial^2 t}{\partial x^2} - \frac{\rho c_p}{k} \frac{\partial t}{\partial \tau} = -\frac{2}{lk}\left(\frac{q}{A}\right) \sin^2 \omega\tau \tag{4}$$

By postulating that the outer surface $(x = 0)$ is insulated and that all the heat is transferred at the inner surface $(x = l)$, the boundary conditions can be written as:

$$\left.\begin{aligned} \frac{\partial t}{\partial x} &= 0 \qquad \text{at} \quad x = 0 \\ \text{and} \qquad & \\ -k\frac{\partial t}{\partial x} &= h(t_l - t_\infty) \qquad \text{at} \quad x = l \end{aligned}\right\} \quad \text{for} \quad \tau > 0$$

The initial condition is obtained by imposing thermal equilibrium at $\tau = 0$. Therefore,

$$t - t_\infty = 0 \qquad \text{at} \quad \tau = 0 \qquad \text{for all} \quad x$$

Introducing the following dimensionless parameters:

$$T^* = \frac{(t - t_\infty)k}{(q/A)l} \qquad x^* = x/l \qquad \tau^* = \omega\tau$$

$$\omega^* = \frac{\omega\rho c_p l^2}{k} \qquad h^* = hl/k$$

and substituting in equation (4) give

$$\frac{\partial^2 T^*}{\partial x^{*2}} - \omega^* \frac{\partial T^*}{\partial \tau^*} = -2 \sin^2\tau^* \tag{5}$$

with boundary conditions

$$\left.\begin{aligned} \frac{\partial T^*}{\partial x^*} &= 0 \qquad \text{at} \quad x^* = 0 \\ \frac{\partial T^*}{\partial x^*} &= -h^*T^* \qquad \text{at} \quad x^* = 1 \end{aligned}\right\} \quad \text{for} \quad \tau^* > 0$$

and initial condition

$$T^* = 0 \qquad \text{at} \quad \tau^* = 0 \qquad \text{for all} \quad x^*$$

As shown in detail in appendix B, the solution is:

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$$T^*(x^*,\tau^*) = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n \cos\lambda_n x^*}{\lambda_n^2\left(\frac{1}{2}\sin 2\lambda_n + \lambda_n\right)}\left[\left(1 - e^{-\lambda_n^2\tau^*/\omega^*}\right)\right.$$

$$\left. -\left(\frac{\lambda_n^4}{\lambda_n^4 + 4\omega^{*2}}\right)\left(\cos 2\tau^* + \frac{2\omega^*}{\lambda_n^2}\sin 2\tau^* - e^{-\lambda_n^2\tau^*/\omega^*}\right)\right] \tag{6}$$

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where λ_n are the eigenvalues of the characteristic equation

$$\lambda \tan \lambda = h^* \tag{7}$$

The first few roots of equation (7) are given in table I for a number of values of h^*.

By means of equation (6), it is thus possible to determine the temperature history at any location in the wall as well as the temperature profile of the wall at any given time.

Applicability of Solution

Consider a typical temperature history curve as shown in figure 1 resulting from equation (6) for an arbitrary set of parameters. This curve clearly indicates the superimposed sinusoidal effect of alternating-current heating on the temperature history obtained in a flat plate with constant heat generation. The time parameter required for the mean temperature parameter to reach 99 percent of the value at $\tau^* = \infty$ is the same for alternating- and direct-current heating. This is also true for the mean value of the temperature parameter in the steady-state region. Hence, to evaluate the mean temperature, an equivalent constant heat generation may be assumed. As shown in appendix B, the solution is

$$T_c^* = \frac{1}{h^*} + \frac{1}{2} - \frac{x^{*2}}{2} - 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n}{\lambda_n^2\left(\frac{1}{2}\sin 2\lambda_n + \lambda_n\right)} e^{-\lambda_n^2\tau^*/\omega^*} \cos\lambda_n x^* \tag{B17}$$

By letting $\tau^* \to \infty$ this solution further reduces to

$$T_c^* = \frac{1}{h^*} + \frac{1}{2} - \frac{x^{*2}}{2} \tag{8}$$

where T_c^* is the temperature parameter resulting from constant heat generation.

To determine the transient temperature in a plate heated by an alternating current, it becomes necessary to use equation (6). Moreover, equation (6) can be used to obtain the steady-state oscillatory part of the temperature history by letting the time parameter τ^* approach infinity, thus eliminating the exponential terms (the transient part) in the solution. Differentiation of this reduced solution yields the minimums and maximums of the steady-state cyclic portion of the temperature history. The differences of these maximum and minimum temperature parameters are then the magnitudes of the uniform temperature oscillations resulting from alternating-current heating.

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RESULTS AND DISCUSSION

Transient Temperature Response

Transient temperature parameters were calculated from equation (6) for several values of h^* and ω^* for both surfaces. The results could not be presented with a single family of curves. Typical transient temperature parameters at the two surfaces, as obtained from equation (6) for a given set of parameters ($h^* = 0.1$, $\omega^* = 1$), are shown in figure 2. The transients are almost identical at both surfaces for small values of h^*, but begin to deviate increasingly from each other as h^* increases.

Magnitudes of Uniform Temperature Oscillations

The magnitudes of the sinusoidal temperature oscillations at the surfaces were calculated by the aforementioned method (letting $\tau^* \to \infty$ in eq. (6)), using an IBM 704 digital computer, for a wide range of dimensionless parameters h^* and ω^*. These magnitudes are expressed in terms of dimensionless temperature differences defined by

$$\Delta T^* = T_{max}^* - T_{min}^* = \frac{(t_{max} - t_{min})k}{(q/A)l} \tag{9}$$

The results are plotted in figure 3. A maximum of six terms was used in evaluating the series solution (eq. (6)), and it was found that the last term used in the series was always less than 0.001 percent of the total temperature difference ΔT^*.

The magnitudes of the temperature oscillations at the insulated surface ($x^* = 0$) are shown in figure 3(a) as a function of frequency and heat-transfer parameters. Figure 3(b) presents similar results for

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the heat-transfer surface ($x^* = 1$). The results agree with those of reference 1, which were obtained by a different mathematical technique. It should be noted that the magnitudes of the temperature oscillations as defined herein by equation (9) are numerically twice the amplitudes of reference 1. It can be seen from figure 3 that the temperature difference parameter reaches an asymptotic value for a given Biot modulus as the frequency parameter becomes small. However, these values are not the same at the two surfaces. The relation between the asymptotic values of the temperature difference parameters at the two surfaces can be obtained empirically from figures 3(a) and (b) and may be expressed by

$$\frac{(\Delta T^*)_{x^*=0}}{(\Delta T^*)_{x^*=1}} = 1 + \frac{h^*}{2} \tag{10}$$

Equation (10) indicates that the magnitudes of the temperature fluctuations do not vary significantly across the plate for small values of h^*. As the Biot modulus h^* increases, the position in the plate (x^* value) becomes increasingly important. Figure 3 also shows that an increase in wall thickness, frequency parameter, or Biot modulus will decrease the temperature difference parameter.

Approximate Solution

If infinite conductivity of the plate is assumed, the problem reduces to that of an ordinary differential equation. The solution, given in reference 2, in dimensionless parameters becomes

$$\Delta T^* = \frac{(t_{max} - t_{min})k}{(q/A)l} = \frac{2}{\sqrt{h^{*2} + 4\omega^{*2}}} \tag{11}$$

This simplified solution is presented in graphical form in figure 4 for a wide range of ω^* and h^*. A comparison of equations (6) and (11) is given in figure 5, which tests the validity of the previous assumption. It is seen from figures 5(a) and (b) that equation (11) will yield satisfactory results for values of $h^* < 0.1$ at all values of ω^*.

Application of Results to Heat-Transfer Studies

In boiling investigations as well as in liquid metals studies, and in "burnout" testing in particular, very high heat fluxes are normally encountered. Since resistive heating is often preferred in the boiling studies, the designer of the experimental apparatus is faced with the selection of alternating or direct current as the source of power. The utilization of direct-current heating could result in higher cost and difficult instrumentation, while the use of alternating current could induce large surface temperature variations.

In table II, the magnitudes of the uniform temperature oscillations at the heat-transfer surface due to alternating-current heating are given in terms of heat flux for some of the more common materials used in boiling studies. The wall thicknesses and the heat-transfer coefficients were selected to cover the range reported in recent boiling-heat-transfer literature. The frequencies listed in the table are those most readily available - 60 and 400 cycles per second. The properties of the materials used in the calculations were taken from reference 4 for 32° F. Selecting the properties at 1000° F results in approximately 20-percent reduction of the magnitudes.

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As an example of the use of table II, consider a nickel test-section wall 0.012 inch thick. If the test section is to be heated by 60-cycle alternating current and a heat flux of 10^7 Btu/(hr)(sq ft), the temperature variation at the boiling surface is 120° F, assuming a heat-transfer coefficient of 8000 Btu/(hr)(sq ft)($^{\circ}$F). This is of the same order of magnitude as the mean wall superheat. By increasing the wall thickness to 0.036 inch, the temperature variation is decreased to 41° F for the same heat flux. By increasing the frequency to 400 cycles per second, this value is further reduced to 5.4° F.

This example illustrates that an increase in frequency or wall thickness will result in smaller temperature oscillations. However, these modifications may not always be feasible, since an increase in wall thickness requires higher currents for the same power input, while the use of higher frequency current may result in excessive cost.

SUMMARY OF RESULTS

The results of this analysis can be summarized as follows:

1. A solution was obtained for complete temperature history and profile of a semi-infinite sinusoidally heated flat plate with constant thermal and electric properties.

2. Computed magnitudes of the uniform temperature oscillations due to a sinusoidal heat source (a.c.) are presented graphically for a wide range of frequency and heat-transfer parameters at both the insulated and the heat-transfer surfaces.

3. Large steady-state temperature oscillations can result in the wall at high heat fluxes due to alternating-current heating. A table

is presented giving the magnitudes of the oscillations at the heat-transfer surface in terms of heat flux for some of the materials most commonly used in boiling-heat-transfer investigations.

4. The magnitudes of the uniform temperature oscillations can be assumed independent of position in the wall, with negligible error, if Biot modulus is less than 0.1. In this range, the assumption of infinite conductivity in the wall is valid. An expression is given relating the magnitudes of the temperature oscillations at the two surfaces of the wall, which indicates that the magnitudes at the heat-transfer surface are always smaller than those at the insulating surface.

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APPENDIX A

SYMBOLS

A	mean heat-transfer area of tube, $\pi(r_o + r_i)L$, sq ft
$C_1, C_2, \ldots, C_n$	integration constants
c_p	specific heat, Btu/(lb)(°F)
F	function defining time-dependent heat generation
h	heat-transfer coefficient, Btu/(hr)(sq ft)(°F)
h^*	heat-transfer parameter, hl/k, or Biot modulus, dimensionless
I_{eff}	effective current, amp
i	instantaneous current, amp
k	thermal conductivity, Btu/(hr)(ft)(°F)
L	length of tube, ft
l	thickness of plate or tube wall, ft
M	designation of a function of x only
N	designation of a function of τ only
q	heat flow, Btu/hr
q'''	heat generation per unit volume, Btu/(hr)(cu ft)
R	electrical resistance, ohms
r_i	inside radius of tube, ft
r_o	outside radius of tube, ft
T^*	temperature parameter defined as $(t - t_\infty)k/(q/A)l$, dimensionless
T_c^*	temperature parameter corresponding to constant-heat-generation case, dimensionless
t	temperature, °F

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t_l	temperature at heat-transfer surface, $^{\circ}F$
t^*	temperature parameter corresponding to steady-state case, $\frac{(t - t_\infty)k}{(q/A)l}$, dimensionless
t_∞	ambient temperature, $^{\circ}F$
$t_{max}-t_{min}$	magnitude of uniform temperature oscillation, $^{\circ}F$
ΔT^*	temperature magnitude parameter, $(T^*_{max} - T^*_{min})$ or $\frac{(t_{max} - t_{min})k}{(q/A)l}$, dimensionless
u^*	function of x^* only
v^*	function of x^* and τ^*
x	distance from insulated surface, ft
x^*	distance parameter defined as x/l, dimensionless
λ_n	positive roots of equation $\lambda \tan \lambda = h^*$
μ	constant
ρ	density, lb/cu ft
τ	time, hr
τ^*	time parameter defined by $\omega\tau$, radians
τ^*_o	arbitrary time parameter, radians
ω	frequency of sinusoidal heat source, radians/hr
ω^*	frequency parameter defined by $\omega\rho c_p l^2/k$, dimensionless

Subscripts:

max	maximum
min	minimum

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APPENDIX B

METHOD AND VERIFICATION OF SOLUTION

Method

As shown in the ANALYSIS section (eq. (5)) the physical problem may be expressed in nondimensional form as

$$\frac{\partial^2 T^*}{\partial x^{*2}} - \omega^* \frac{\partial T^*}{\partial t^*} = -2 \sin^2 \tau^* \tag{B1}$$

with the boundary conditions:

$$\left.\begin{aligned} \frac{\partial T^*}{\partial x^*} &= 0 \qquad \text{at} \quad x^* = 0 \\ \frac{\partial T^*}{\partial x^*} &= -h^* T^* \qquad \text{at} \quad x^* = 1 \end{aligned}\right\} \quad \text{at} \quad \tau^* > 0$$

and initial condition

$$T^* = 0 \qquad \text{at} \quad \tau^* = 0 \qquad \text{for all} \quad x^*$$

As the first step in solving equation (B1), its right side will be made constant. This is equivalent to the physical case of constant heat generation. A solution to this simplified equation will then be obtained by assuming a sum of steady-state and transient (time-dependent) solutions. Finally, by applying Duhamel's theorem an expression for T^* will be derived, which will satisfy equation (B1) and its boundary and initial conditions.

Thus, assuming constant heat generation, equation (B1) becomes

$$\frac{\partial^2 T_c^*}{\partial x^{*2}} - \omega^* \frac{\partial T_c^*}{\partial t^*} = -1 \tag{B2}$$

with boundary and initial conditions unchanged and where T_c^* is the temperature parameter for the constant-heat-generation case.

For steady state, equation (B2) reduces to

$$\frac{d^2 t^*}{dx^{*2}} = -1 \tag{B3}$$

with

$$\frac{dt^*}{dx^*} = 0 \qquad \text{at} \quad x^* = 0$$

and

$$\frac{dt^*}{dx^*} = -h^*t^* \qquad \text{at} \quad x^* = 1$$

where t^* is a dimensionless temperature for the steady-state case. The solution of equation (B3) is

$$t^* = \frac{1}{2h^*}\left(2 + h^* - h^*x^{*2}\right) \tag{B4}$$

As suggested in reference 5, let the solution of equation (B2) be of the form

$$T_c^* = u^* + v^* \tag{B5}$$

where u^* and v^* are the steady-state and transient solutions, respectively. Equation (B5) then becomes

$$T_c^* = \frac{1}{2h^*}\left(2 + h^* - h^*x^{*2}\right) + v^* \tag{B6}$$

Substituting into equation (B2) yields

$$\frac{\partial^2 v^*}{\partial x^{*2}} - \omega^* \frac{\partial v^*}{\partial \tau^*} = 0 \tag{B7}$$

with the boundary conditions

$$\frac{\partial v^*}{\partial x^*} = 0 \qquad \text{at} \quad x^* = 0$$

and

$$\frac{\partial v^*}{\partial x^*} = -h^*v^* \qquad \text{at} \quad x^* = 1$$

and the initial condition

$$v^* = -\frac{1}{2h^*}\left(2 + h^* - h^*x^{*2}\right) \qquad \text{at} \quad \tau^* = 0$$

A solution for v^* from equation (B7) may be obtained by separating the variables. Let

$$v^*(x^*,\tau^*) = M(x^*)N(\tau^*)$$

Therefore,

$$M''(x^*)N(\tau^*) - \omega^* M(x^*)N'(\tau^*) = 0$$

where the primes denote the derivatives; or,

$$\frac{M''(x^*)}{M(x^*)} = \omega^* \frac{N'(\tau^*)}{N(\tau^*)} = \text{constant} = \mu$$

If μ were positive, the temperature of the plate would become infinite with time. If μ were zero, then the function expressing the time dependence of the temperature would be a constant. Since both of these possibilities violate the physical conditions of the problem, μ must be negative. For convenience μ is defined:

$$\mu = -\lambda^2$$

Hence,

$$M''(x^*) + \lambda^2 M(x^*) = 0$$

and

$$N'(\tau^*) + \frac{\lambda^2}{\omega^*} N(\tau^*) = 0$$

or

$$\left.\begin{aligned} M(x^*) &= C_1 \cos \lambda x^* + C_2 \sin \lambda x^* \\ \text{and} \qquad & \\ N(\tau^*) &= C_3 e^{-\lambda^2 \tau^*/\omega^*} \end{aligned}\right\} \tag{B8}$$

$$M'(x^*) = -\lambda C_1 \sin \lambda x^* + \lambda C_2 \cos \lambda x^* \tag{B9}$$

From the first boundary condition of equation (B1) it can be noted that

$$M'(0) = 0 = \lambda C_2$$

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But,

$$\lambda \neq 0$$

Therefore,

$$C_2 = 0$$

Thus using equation (B8) and letting $C_4 = C_1C_3$ result in

$$v^*(x^*,\tau^*) = M(x^*)N(\tau^*) = C_4 e^{-\lambda^2\tau^*/\omega^*} \cos x^* \tag{B10}$$

The second boundary condition of equation (B1) yields:

$$M'(1)N(\tau^*) = -h^*v^*(1,\tau^*)$$

Substituting equation (B6) for $T_c^*(1)$ yields

$$M'(1)N(\tau^*) = -h^*v^*(1,\tau^*)$$

But from equation (B10),

$$v^*(1,\tau^*) = M(1)N(\tau^*)$$

Hence,

$$M'(1) = -h^*M(1)$$

Noting that $C_2 = 0$ and evaluating this by using equations (B8) and (B9) for $x^* = 1$ yield

$$\lambda \tan \lambda = h^* \tag{B11}$$

Equation (B11) has an infinite number of roots. The roots sufficient for most computations are given in table I for a number of values of h^*. It was shown previously that $\mu \neq 0$; hence, a value of $\lambda = 0$ may be disregarded. Thus, equation (B10) may be rewritten as

$$v^*(x^*,\tau^*) = \sum_{n=1}^{\infty} C_n e^{-\lambda_n^2\tau^*/\omega^*} \cos \lambda_n x^* \tag{B12}$$

where λ_n designates all the positive roots of equation (B11). The constants in equation (B12) are obtained as follows:

Using the initial condition in equation (B12) gives

$$v^*(x^*,0) = \sum_{n=1}^{\infty} C_n \cos \lambda_n x^* = -\frac{1}{2h^*}\left(2 + h^* - h^* x^{*2}\right) \qquad \text{(B13)}$$

Multiplying both sides by $\cos \lambda_j x^*$, integrating over the interval $0 \leq x^* \leq 1$, and interchanging the order of integration and summation yield

$$\sum_{n=1}^{\infty} C_n \int_0^1 \cos \lambda_n x^* \cos \lambda_j x^* \, dx^*$$

$$= -\frac{1}{2h^*} \int_0^1 \left(2 + h^* - h^* x^{*2}\right) \cos \lambda_j x^* \, dx^* \qquad \text{(B14)}$$

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Integrating by parts twice shows that

$$\int_0^1 \cos \lambda_j x^* \cos \lambda_n x^* \, dx^* = 0 \qquad \text{if} \quad \lambda_n \neq \lambda_j$$

Therefore, all the terms in the series are zero except for $n = j$; hence (B14) becomes

$$C_j \int_0^1 \cos^2 \lambda_j x^* \, dx^* = -\frac{1}{2h^*} \int_0^1 \left(2 + h^* - h^* x^{*2}\right) \cos \lambda_j x^* \, dx^*$$

Integration, rearrangement, and use of equation (B11) yield

$$C_j = -\frac{2 \sin \lambda_j}{\lambda_j^2 \left(\frac{1}{2} \sin 2\lambda_j + \lambda_j\right)} \qquad \text{(B15)}$$

Combining equation (B15) with equation (B12) gives

$$v^*(x^*,\tau^*) = -2 \sum_{n=1}^{\infty} \frac{\sin \lambda_n}{\lambda_n^2 \left(\frac{1}{2} \sin 2\lambda_n + \lambda_n\right)} e^{-\lambda_n^2 \tau^*/\omega^*} \cos \lambda_n x^* \qquad \text{(B16)}$$

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By substituting equation (B16) in equation (B6) an expression for T_c^* is obtained. Thus,

$$T_c^* = \frac{1}{h^*} + \frac{1}{2} - \frac{x^{*2}}{2} - 2 \sum_{n=1}^{\infty} \frac{\sin \lambda_n}{\lambda_n^2\left(\frac{1}{2} \sin 2\lambda_n + \lambda_n\right)} e^{-\lambda_n^2 \tau^*/\omega^*} \cos \lambda_n x^* \tag{B17}$$

Equation (B17) is the solution for the unsteady case with constant heat generation represented by equation (B2). Since the heat generation is a function of time, say $F(\tau^*)$, a solution to equation (B1) can be derived by the use of Duhamel's theorem, which implies:

$$T^*(x^*,\tau^*) = -\int_0^{\tau^*} F(\tau_o^*) \frac{\partial T_c^*(x^*,\tau^* - \tau_o^*)}{\partial \tau^*} \partial \tau_o^* \tag{B18}$$

From equation (B1),

$$F(\tau_o^*) = -2 \sin^2 \tau_o^* = \cos 2\tau_o^* - 1 \tag{B19}$$

and from equation (B17),

$$\frac{\partial T_c^*(x^*,\tau^* - \tau_o^*)}{\partial \tau^*} = \frac{2}{\omega^*} \sum_{n=1}^{\infty} \frac{\sin \lambda_n \cos \lambda_n x^*}{\frac{1}{2} \sin 2\lambda_n + \lambda_n} e^{-\lambda_n^2(\tau^*-\tau_o^*)/\omega^*} \tag{B20}$$

Substituting equations (B19) and (B20) into equation (B18) gives

$$T^*(x^*,\tau^*) = \frac{2}{\omega^*} \sum_{n=1}^{\infty} \int_0^{\tau^*} (1 - \cos 2\tau_o^*) \frac{\sin \lambda_n \cos \lambda_n x^*}{\frac{1}{2} \sin 2\lambda_n + \lambda_n} e^{-\lambda_n^2(\tau^*-\tau_o^*)/\omega^*} d\tau_o^*$$

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Integrating and rearranging finally yield

$$T^*(x^*,\tau^*) = 2\sum_{n=1}^{\infty} \frac{\sin\lambda_n \cos\lambda_n x^*}{\lambda_n^2\left(\frac{1}{2}\sin 2\lambda_n + \lambda_n\right)}\left[\left(1 - e^{-\lambda_n^2\tau^*/\omega^*}\right) - \left(\frac{\lambda_n^4}{\lambda_n^4 + 4\omega^{*2}}\right)\left(\cos 2\tau^* + \frac{2\omega^*}{\lambda_n^2}\sin 2\tau^* - e^{-\lambda_n^2\tau^*/\omega^*}\right)\right] \tag{B21}$$

Equation (B21) is a series solution of the original differential equation (B1). The eigenvalues λ_n can be determined from the characteristic equation $\lambda \tan \lambda = h^*$.

Verification

Repeated differentiation of equation (B21), substitution into equation (5), and some algebraic manipulation give

$$\frac{1}{2} = \sum_{n=1}^{\infty} \frac{\sin\lambda_n}{\frac{1}{2}\sin 2\lambda_n + \lambda_n}\cos\lambda_n x^* \tag{B22}$$

Hence, equation (B21) satisfies the given differential equation (5) if, and only if, the right side of equation (B22) converges to 1/2. The appearance of eigenvalues makes the analytical proof of convergence difficult. However, an IBM 704 digital computer was utilized to evaluate the series terms of equation (B22) for a number of values of h^* and x^*. For all cases tested the right-side term did converge to 1/2, indicating that equation (B22) is an identity.

To show that equation (B21) satisfies the boundary conditions, differentiate it, which will result in

$$\frac{\partial T^*}{\partial x^*} = 2\sum_{n=1}^{\infty} \frac{-\lambda_n \sin\lambda_n \sin\lambda_n x^*}{\lambda_n^2\left(\frac{1}{2}\sin 2\lambda_n + \lambda_n\right)}\left[\left(1 - e^{-\lambda_n^2\tau^*/\omega^*}\right) - \left(\frac{\lambda_n^4}{\lambda_n^4 + 4\omega^{*2}}\right)\left(\cos 2\tau^* + \frac{2\omega^*}{\lambda_n^2}\sin 2\tau^* - e^{-\lambda_n^2\tau^*/\omega^*}\right)\right] \tag{B23}$$

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It follows from equation (B23) that

$$\frac{\partial T^*}{\partial x^*} = 0 \qquad \text{at} \quad x^* = 0 \qquad \text{for} \quad \tau^* > 0$$

Evaluating equation (B23) at $x^* = 1$ gives

$$\frac{\partial T^*(1,\tau^*)}{\partial x^*} = -2 \sum_{n=1}^{\infty} \frac{\lambda_n \tan \lambda_n \sin \lambda_n \cos \lambda_n}{\lambda_n^2\left(\frac{1}{2} \sin 2\lambda_n + \lambda_n\right)} \left[\left(1 - e^{-\lambda_n^2 \tau^*/\omega^*}\right) - \left(\frac{\lambda_n^4}{\lambda_n^4 + 4\omega^{*2}}\right)\left(\cos 2\tau^* + \frac{2\omega^*}{\lambda_n^2} \sin 2\tau^* - e^{-\lambda_n^2 \tau^*/\omega^*}\right)\right] \tag{B24}$$

Recalling that $h^* = \lambda \tan \lambda$, equation (B24) reduces to

$$\frac{\partial T^*}{\partial x^*} = -h^*T^* \qquad \text{at} \quad x^* = 1 \qquad \text{for} \quad \tau^* > 0$$

By inspection of equation (B21), it is evident that

$$T^* = 0 \qquad \text{at} \quad \tau^* = 0 \qquad \text{for all} \quad x^*$$

Thus, equation (B21) satisfies the given differential equation along with the boundary and initial conditions.

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TABLE I. - ROOTS OF EQUATION: $h^* = \lambda \tan \lambda$

h^*	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
0.001	0.03162	-------	-------	-------	--------	--------
.002	.04471	-------	-------	-------	--------	--------
.005	.07065	3.14318	-------	-------	--------	--------
.01	.99830	3.14477	6.28478	-------	--------	--------
.02	.14095	3.14795	6.28637	9.42690	12.56796	--------
.05	.22176	3.15743	6.29113	9.43008	12.57035	15.71115
.1	.31105	3.17310	6.29906	9.43538	12.57432	15.71433
.2	.43284	3.20393	6.31485	9.44595	12.58226	15.72068
.5	.65327	3.29231	6.36162	9.47748	12.60601	15.73972
1.0	.86033	3.42562	6.43730	9.52933	12.64529	15.77128
2.0	1.07687	3.64360	6.57833	9.62956	12.72230	15.83361
5.0	1.31384	4.03357	6.90960	9.89275	12.93522	16.01066
10.0	1.42887	4.30580	7.22811	10.20026	13.21418	16.25936
20.0	1.49613	4.49148	7.49541	10.51167	13.54198	16.58640
∞	1.57080	4.71239	7.85399	10.99557	14.13717	17.27876

TABLE II. - MAGNITUDE OF UNIFORM TEMPERATURE OSCILLATIONS AT THE HEAT-TRANSFER SURFACE IN TERMS OF HEAT FLUX FOR SOME COMMON MATERIALS USED IN BOILING STUDIES

Material (a)	Heat-transfer coefficient, h, Btu/(hr)(sq ft)(°F)	Plate thickness, in.									
		0.003		0.006		0.012		0.036		0.072	
		Frequency, cps		Frequency, cps		Frequency, cps		Frequency, cps		Frequency, cps	
		60	400	60	400	60	400	60	400	60	400
		Magnitude of temperature oscillations in terms of heat flux $(t_{max} - t_{min})/(q/A)$, °F/(Btu)(hr)(sq ft)									
Stainless 347	2,000	56x10⁻⁶	7.0x10⁻⁶	28x10⁻⁶	3.4x10⁻⁶	14x10⁻⁶	1.8x10⁻⁶	4.5x10⁻⁶	0.58x10⁻⁶	2.3x10⁻⁶	
	8,000	50	6.9	24	3.4	12	1.8	4.2	.56	2.2	
	20,000	41	6.2	20	3.1	9.3	1.6	3.5	.49	1.6	
Nickel	2,000	51x10⁻⁶	6.5x10⁻⁶	26x10⁻⁶	3.2x10⁻⁶	13x10⁻⁶	1.6x10⁻⁶	4.3x10⁻⁶	0.54x10⁻⁶	2.1x10⁻⁶	
	8,000	50	6.5	26	3.2	12	1.6	4.1	.54	2.1	
	20,000	44	6.4	23	3.1	10	1.5	3.6	.53	1.7	
Brass (70% Cu, 30% Zn)	2,000	60x10⁻⁶	7.4x10⁻⁶	30x10⁻⁶	3.8x10⁻⁶	15x10⁻⁶	1.9x10⁻⁶	5.0x10⁻⁶	0.62x10⁻⁶	2.5x10⁻⁶	
	8,000	58	7.4	30	3.8	15	1.9	4.8	.62	2.4	
	20,000	53	7.4	27	3.6	13	1.8	4.3	.61	2.3	
Aluminum	2,000	84x10⁻⁶	11x10⁻⁶	43x10⁻⁶	5.2x10⁻⁶	21x10⁻⁶	2.7x10⁻⁶	7.0x10⁻⁶	0.88x10⁻⁶	3.5x10⁻⁶	0.43x10⁻⁶
	8,000	79	11	43	5.2	21	2.7	6.7	.88	3.4	.43
	20,000	62	11	38	5.2	19	2.6	6.2	.85	3.0	.43
Copper	2,000	54x10⁻⁶	6.5x10⁻⁶	27x10⁻⁶	3.3x10⁻⁶	14x10⁻⁶	1.6x10⁻⁶	4.6x10⁻⁶	0.55x10⁻⁶	2.2x10⁻⁶	0.27x10⁻⁶
	8,000	53	6.5	27	3.3	14	1.6	4.4	.55	2.2	.27
	20,000	46	6.5	25	3.3	13	1.6	4.2	.55	2.1	.27

[a]Properties are taken at 32° F from ref. 4.

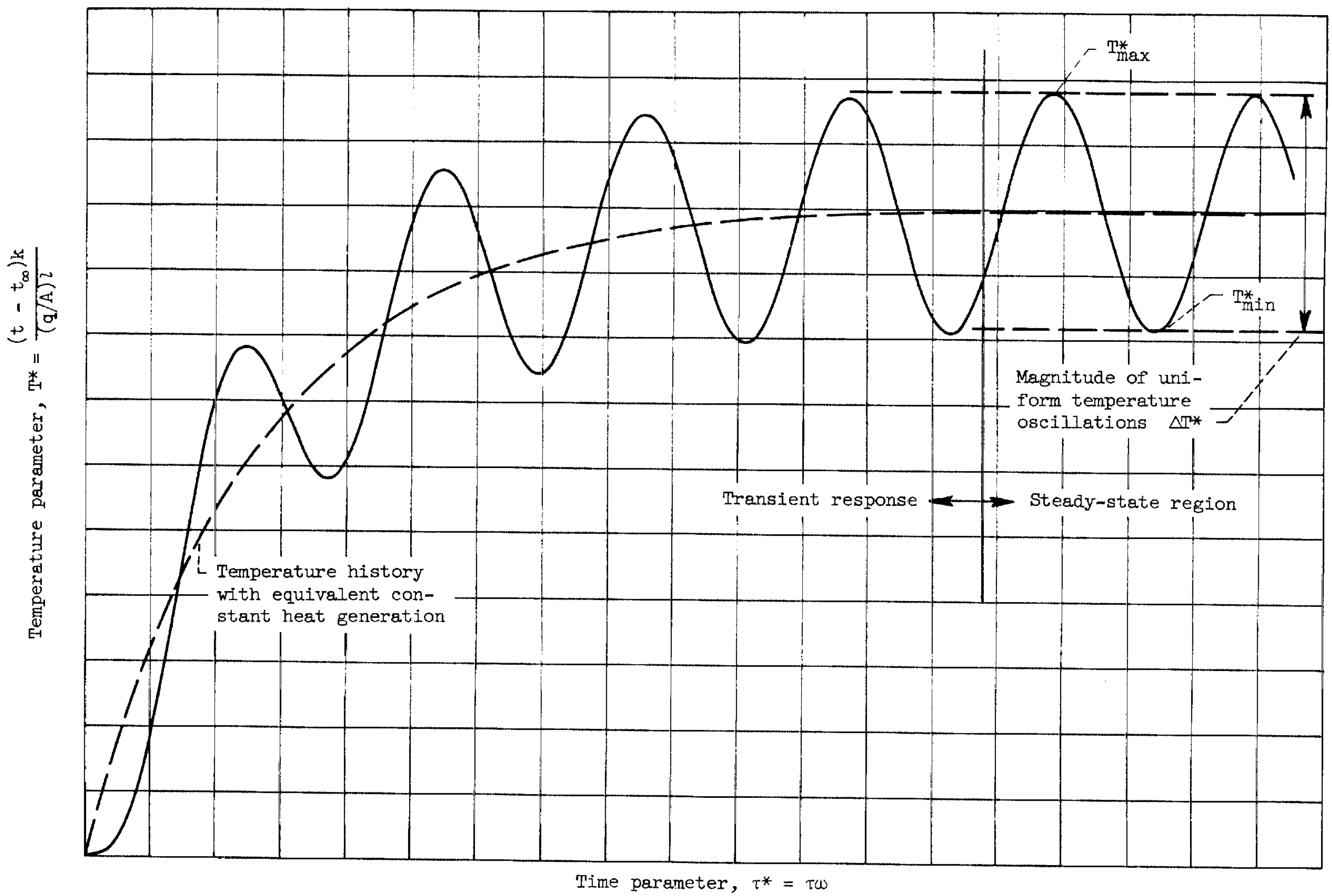


Figure 1. - Typical temperature history of a sinusoidally heated flat plate.

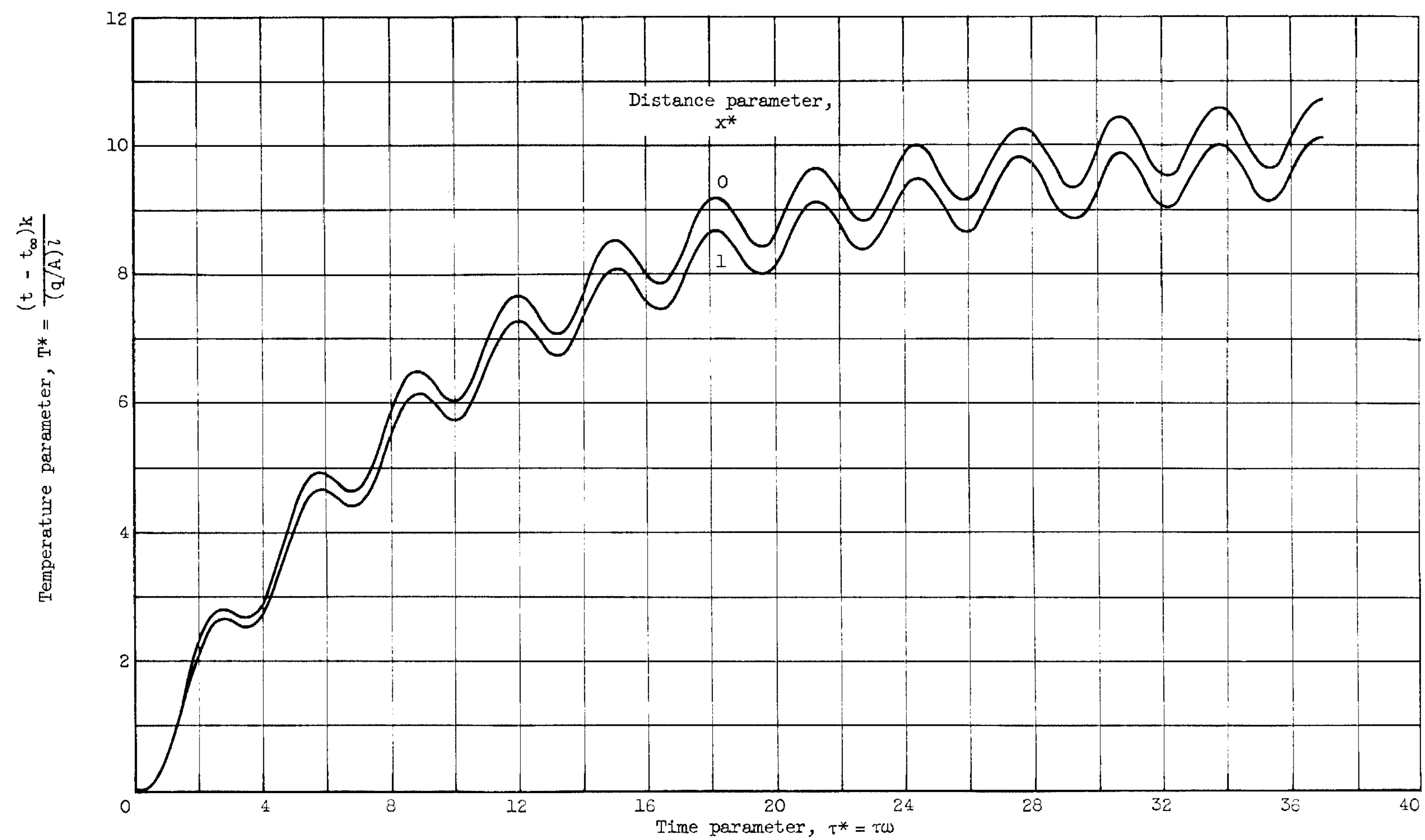
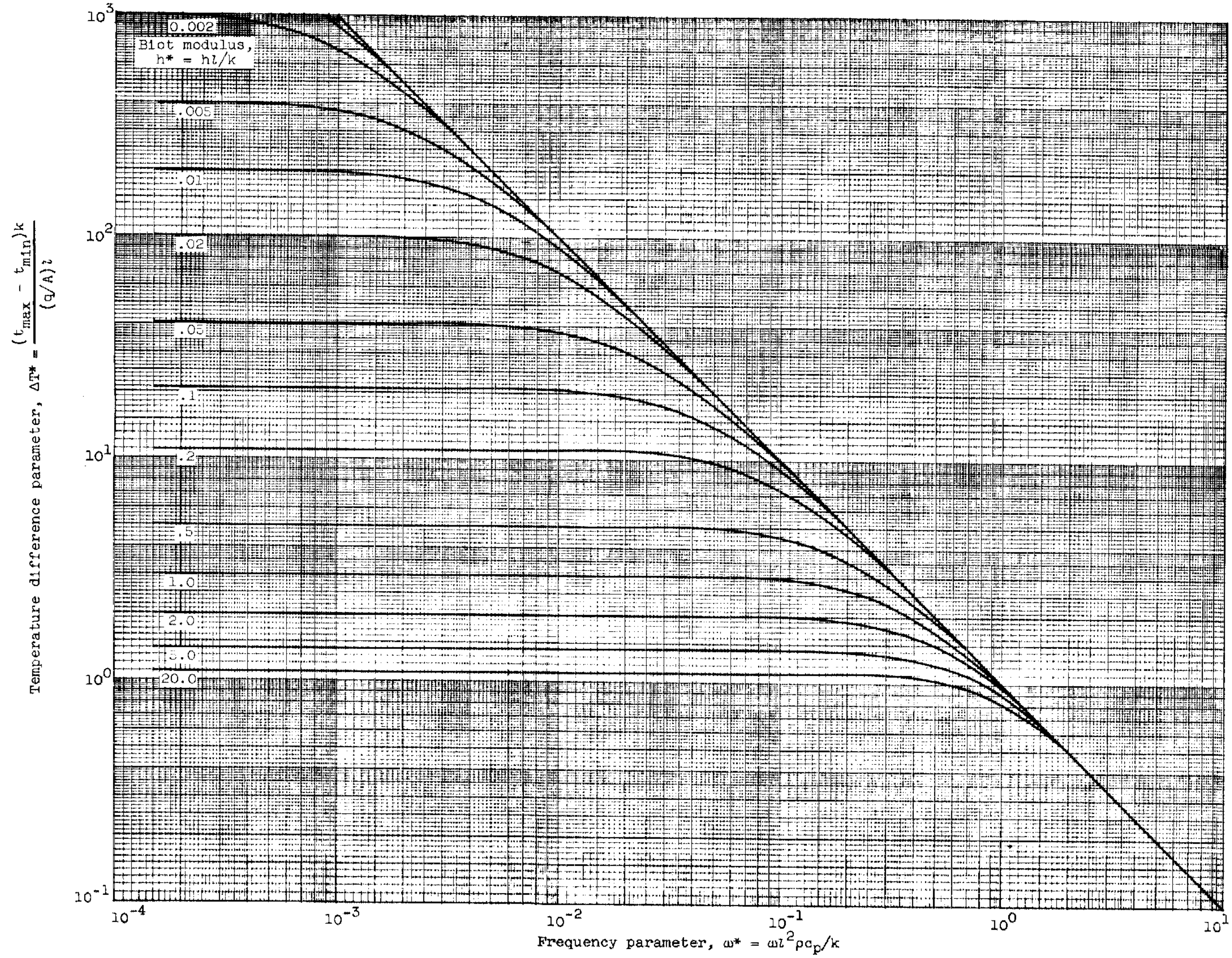
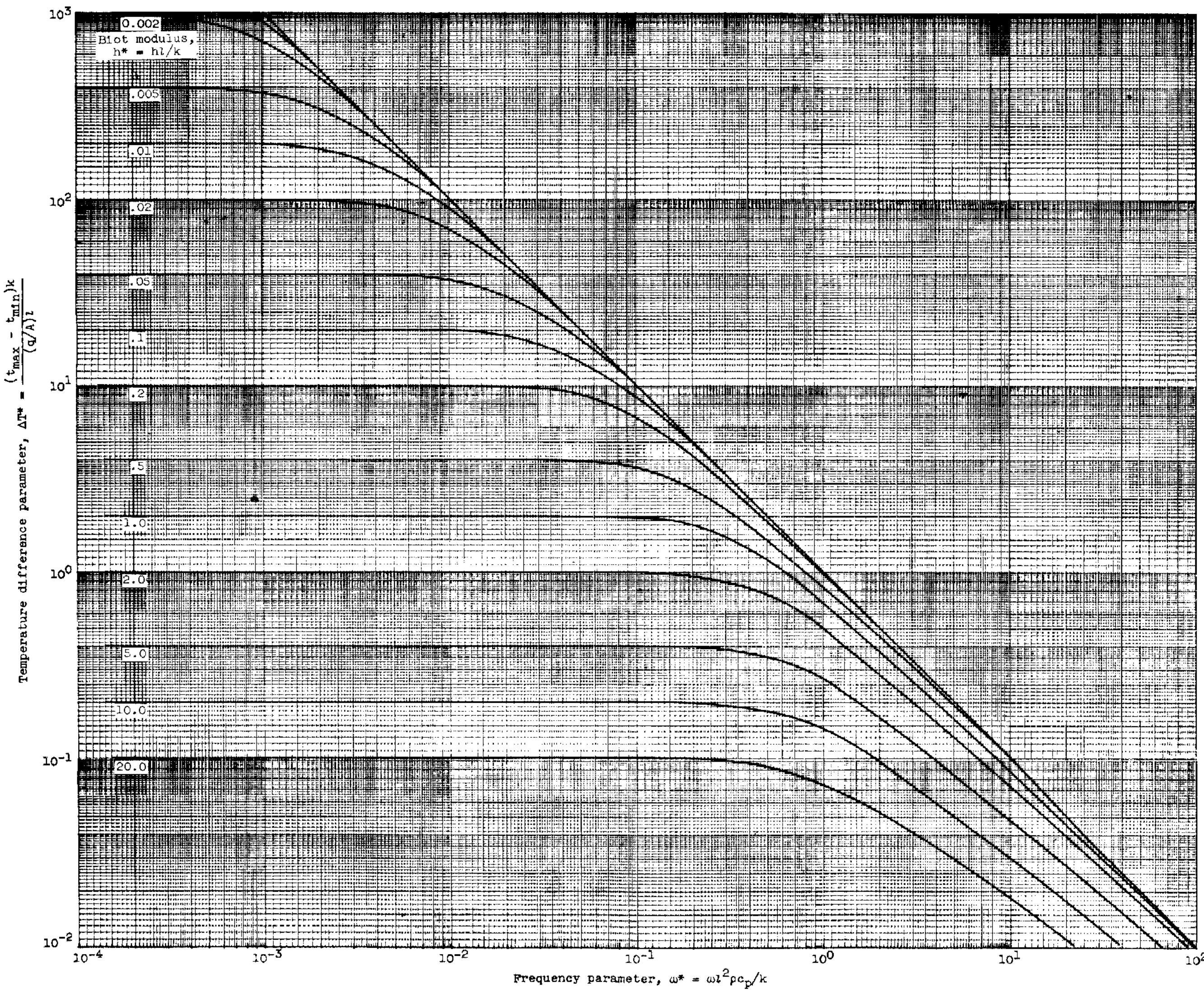


Figure 2. - Transient response at the two surfaces of a sinusoidally heated plate for $h^* = 0.1$ and $\omega^* = 1.0$.



(a) Insulated surface.

Figure 3. - Magnitude of uniform temperature oscillations at surfaces of a sinusoidally heated plate.



(b) Heat-transfer surface.

Figure 3. - Concluded. Magnitude of uniform temperature oscillations at surfaces of a sinusoidally heated plate.

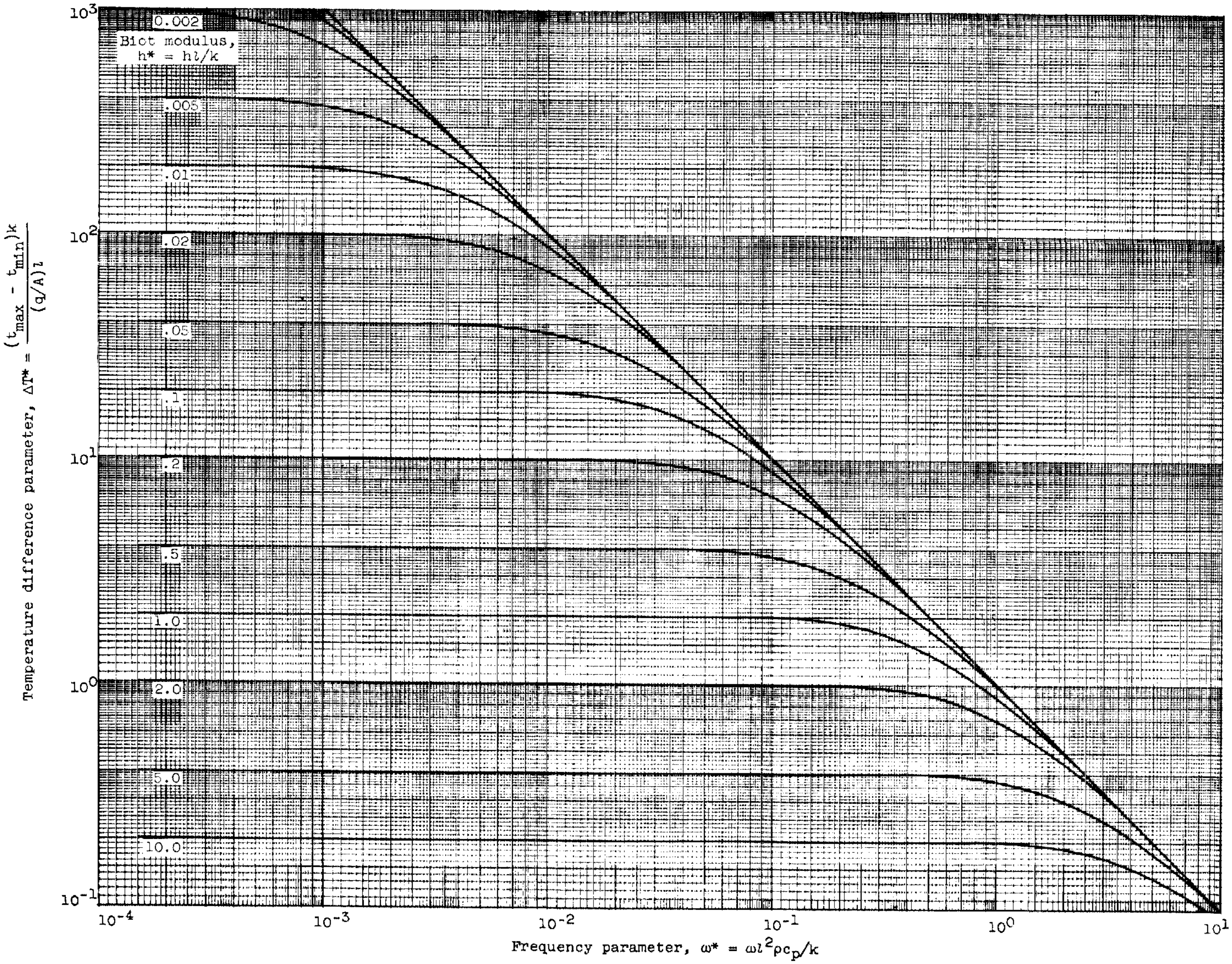
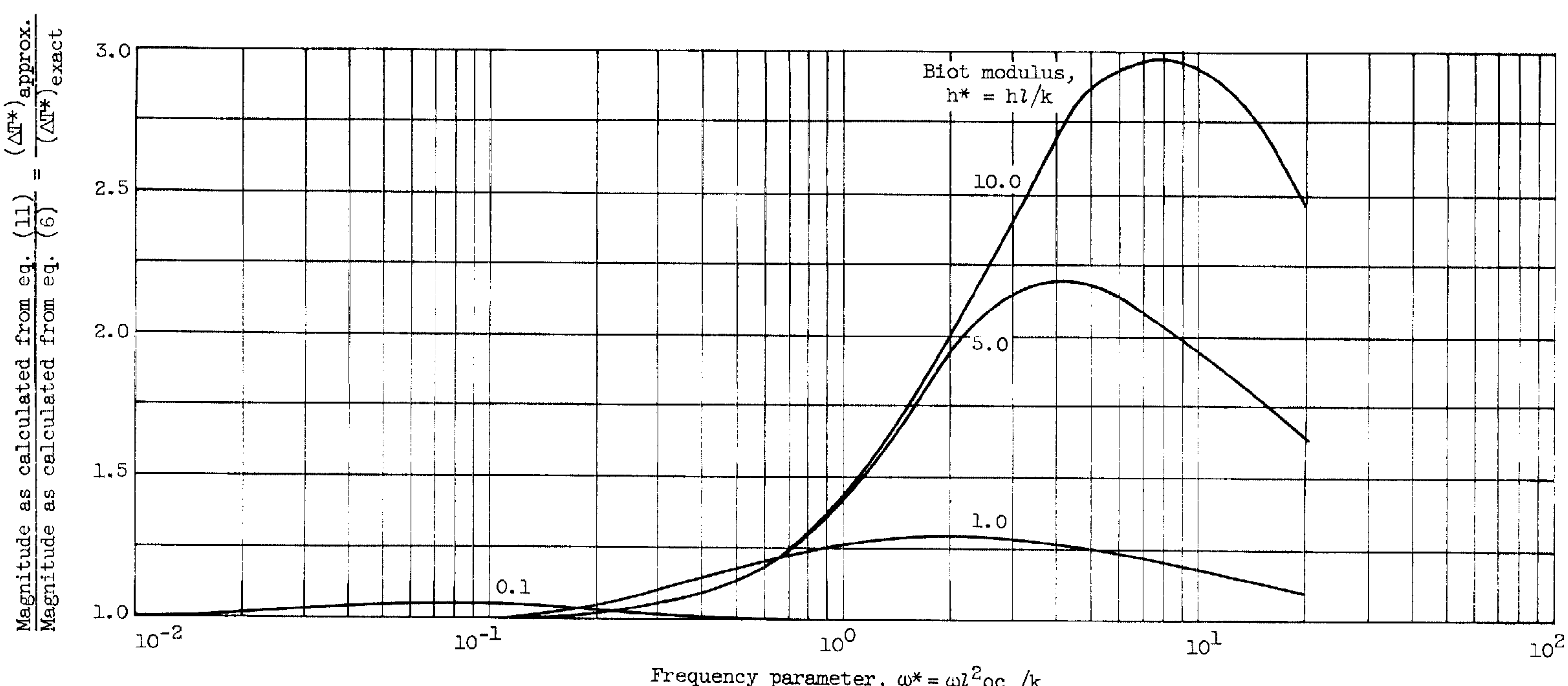
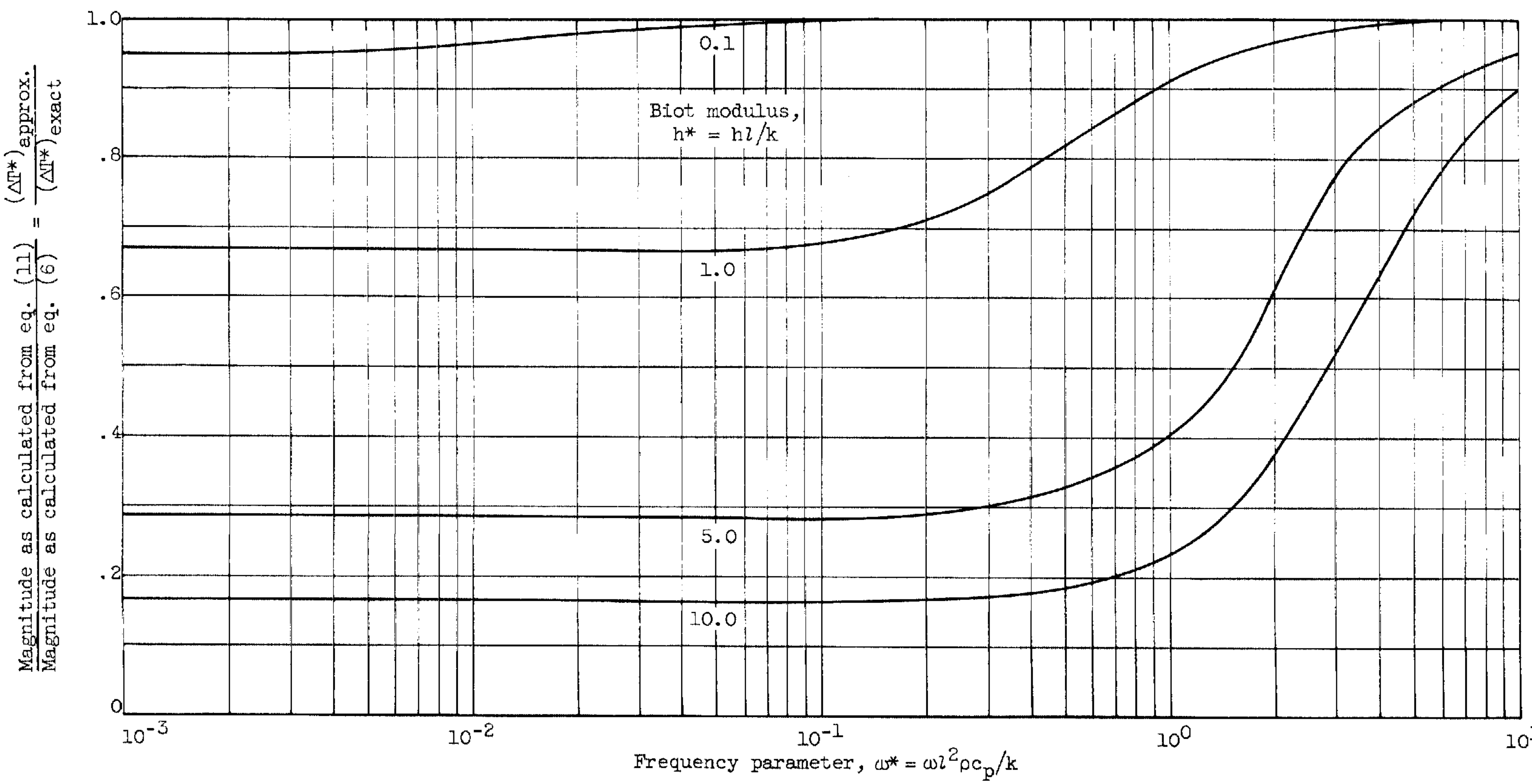


Figure 4. - Magnitude of uniform temperature oscillations in a sinusoidally heated flat plate assuming infinite conductivity.



(a) Heat-transfer surface.

Figure 5. - Error obtained in calculating magnitudes of temperature oscillations at surfaces by assuming infinite wall conductivity.



(b) Insulated surface.

Figure 5. - Concluded. Error obtained in calculating magnitudes of temperature oscillations at surfaces by assuming infinite wall conductivity.

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National Aeronautics and Space Administration.
AN ANALYTICAL DETERMINATION OF TEMPERATURE OSCILLATIONS IN A WALL HEATED BY ALTERNATING CURRENT. Frank A. Jeglic. July 1962. 28p. OTS price, $0.75.
(NASA TECHNICAL NOTE D-1286)

A solution is obtained for a complete temperature history and profile of a sinusoidally heated plate with constant thermal and electrical properties and one insulated surface. The analysis is valid for thin, long tubes. The magnitudes of the uniform temperature oscillations at the surfaces of the plate, caused by the sinusoidal heat source, are presented in a graphical form in terms of frequency and heat-transfer parameters.

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